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k-string tensions in the 3-d $\mathrm{SU}(N)$ Georgi-Glashow model

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Abstract: The classic argument by Polyakov showing that monopoles produce confinement in the Higgs phase of the Georgi-Glashow model is generalized to study the spectrum of k-strings. We find that the leading-order low-density approximation yields Casimir scaling in the weakly-coupled 3-d $\mathrm{SU}(N)$ Georgi-Glashow model. Corrections to the Casimir formula are considered. When $k \sim N$, the non-diluteness effect is of the same order as the leading term, indicating that non-diluteness can significantly change the Casimir-scaling behavior. The correction produced by the propagating Higgs field is also studied and found to increase, together with the non-diluteness effect, the Casimir-scaling ratio. Furthermore, a correction due to closed k-strings is also computed and is shown to yield the same k-dependence as the one due to non-diluteness, but with the opposite sign and a nontrivial N-dependence. Finally, we consider the possible implications of our analysis for the $\mathrm{SU}(N)$ analogue of compact QED in four dimensions.

Keywords: Field Theories in Lower Dimensions, Nonperturbative Effects, Confinement, Lattice Gauge Field Theories.

This work is dedicated to the memory of Ian Kogan.

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1. Introduction

The current description of strong interactions is based on QCD, whose dynamics becomes nonperturbative at energies of order Λ_{QCD}. Confinement, which can be defined as the absence in the observed spectrum of asymptotic states that carry a color charge, is commonly associated to a nonperturbative, linearly rising potential between color sources - for recent discussions on confinement see e.g. [1, 2] and references therein. Such a potential is indeed consistent with the Regge trajectories observed in the hadronic spectrum and hints to a picture where hadrons are made of quarks joined by flux tubes of chromoelectric field. String theory was originally introduced as an effective theory to describe the dynamics of these flux tubes. Numerical simulations of gauge theories regulated on a lattice are an effective tool to investigate nonperturbative properties from first principles; they have confirmed the existence of such a linear potential, have provided good evidence in favor of a bosonic string description, and have studied in some details the structure of the flux tubes.

While this picture is generally accepted nowadays, the mechanism which is responsible for confinement is still under active debate. New aspects of the mechanism of confinement may be evidentiated by studying SU(N) gauge theories for $N > 3$, under the assumption that all these theories share the same basic properties with corrections that are organised in powers of $1/N$. These ideas have triggered a recent interest in the spectrum of k-strings in SU(N) gauge theories. A k-string is defined as the confining flux tube between sources in higher representations, carrying a charge k with respect to the center of the gauge group Z_N, i.e. representations with nonvanishing N-ality. These sources can be seen as the superposition of k fundamental charges, and charge conjugation exchanges k- and $(N-k)$-strings, so that non trivial k-strings exist only for $N > 3$;[1] their string tensions σ_k can be - and should be - used to constrain mechanisms of confinement [3, 4]. Results for the

[1] We shall not consider in this work high-dimensional representations that are screened by gluons and do not yield a genuine asymptotic string tension.

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values of σ_k can be obtained by various approaches. Early results, based on dimensional reduction arguments, suggest the so-called "Casimir scaling" hypothesis for the ratio of string tensions [5]:

$$R(k,N) \equiv \frac{\sigma_k}{\sigma_1} = \frac{k(N-k)}{N-1} \equiv C(k,N) \tag{1.1}$$

where σ_1 is the fundamental string tension. Recent studies in supersymmetric Yang-Mills theories and M-theory suggest instead a "Sine scaling" formula:

$$R(k,N) = \frac{\sin(k\pi/N)}{\sin(\pi/N)}\,. \tag{1.2}$$

Corrections are expected to both formulae, but the form of such corrections is unknown for the physically relevant case of a four dimensional, nonsupersymmetric, SU(N) gauge theory.

In the large-N limit, where the interactions between flux tubes are suppressed by powers of $1/N$, the lowest energy state of the system should be made of k fundamental flux tubes connecting the sources; hence:

$$R(k,N) \overset{k-\text{fixed}}{\underset{}{\xrightarrow{N\to\infty}}} k\,. \tag{1.3}$$

Both the Casimir and the Sine scaling formulae satisfy this constraint; they also remain invariant under the replacement $k \to (N-k)$, which corresponds to the exchange of quarks with antiquarks. However, it has been argued in refs. [6, 7] that the corrections to the large-N behavior should occur as a power series in $1/N^2$ rather than $1/N$.[2] Clearly such a behavior would exclude Casimir scaling as an *exact* description of the k-string spectrum.

Recent lattice calculations have provided new results for the spectrum of k-strings both in three and four dimensions [8]–[13]. They all confirm that Casimir scaling is a *good approximation* to the Yang-Mills results. To be more quantitative, one could say that all lattice results are within 10% of the Casimir scaling prediction, and that deviations from it are larger in four than they are in three dimensions, in agreement with strong-coupling predictions [12]. The taming of systematic errors is a crucial matter for such lattice calculations, and it can only be achieved by an intensive numerical analysis. In four dimensions, the higher statistics simulations presented in ref. [12] show that corrections to the Casimir scaling formula are statistically significant, and actually favor the Sine scaling. Finally, it has been pointed out in ref. [13] that higher-dimensional representations with common N-ality do yield the same string tension, as expected because of gluon screening.

These numerical results trigger a few comments on Casimir scaling. The original argument [5] was based on the idea that a 4-d gauge theory in a random magnetic field could be described by a 2-d theory without such a field. Besides the numerical results, there is little support for such an argument in QCD; moreover it is not clear that the same hypothesis could explain the approximate Casimir scaling observed in three dimensions. On the other hand, Casimir scaling appears "naturally" as the lowest order result, both at

[2] One should study with some care whether the arguments presented in refs. [6, 7] hold independently of the space-time dimensionality.

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strong-coupling in the case of k-strings in the hamiltonian formulation of gauge theories, and in the case of the spectrum of bound states in chiral models. Corrections can be computed in the strong-coupling formulation and they turn out to be $\propto (d-2)/N$ - see e.g. ref. [12] for a summary of results and references. While we are aware that strong-coupling calculations are not directly relevant to describe the physics of the continuum theory, we think that it is nonetheless instructive to have some quantitative analytic control within that framework. Last but not least, Casimir scaling also appears at the lowest order in the stochastic model of the QCD vacuum [14]. In view of these considerations, it is fair to say that approximate Casimir scaling should be a prerequisite for any model of confinement, that corrections should be expected, and that these corrections are liable to yield further informations about the non-perturbative dynamics of strong interactions. Moreover, it would be very interesting to improve our understanding of some other aspects of the k-string spectrum, like e.g. the origin of the Sine scaling for non-supersymmetric theories, or the structure of the corrections to this scaling form.

In order to get more insight in the dynamics underlying the k-string spectrum, we compute in this paper the ratio $R(k, N)$ in the 3-d SU(N) Georgi-Glashow model. This model, in the ($N = 2$)-case, is a classic example [15] of a theory, which allows for an analytic description of confinement. The latter is due to the plasma of point-like magnetic monopoles, which produce random magnetic fluxes through the contour of the Wilson loop. In the weak-coupling regime of the model (which is assumed henceforth), this plasma is dilute, and the interaction between monopoles is Coulombic, being induced by the dual-photon exchanges. Since the energy of a single monopole is a quadratic function of its flux, it is energetically favorable for the vacuum to support a configuration of two monopoles of unit charge (in the units of the magnetic coupling constant, g_m), rather than a single, doubly-charged monopole. Owing to this fact, monopoles of unit charge dominate in the vacuum, whereas monopoles with higher charges tend to dissociate into them. Summing over the grand canonical ensemble of monopoles of unit charge, interacting with each other by the Coulomb law, one arrives at an effective low-energy theory, which is a 3-d sine-Gordon theory of a dual photon. The latter acquires a mass (visible upon the expansion of the cosine potential) by means of the Debye screening in the Coulomb plasma. The appearance of this (exponentially small) mass and, hence, of a finite (albeit exponentially large) vacuum correlation length is crucial for the generation of the fundamental string tension, i.e., for the confinement of the external fundamental matter. It is worth noting that a physically important interpretation of these ideas in terms of spontaneous breaking of magnetic Z_2 symmetry has been presented in reviews [16] and refs. therein.

In the next section, we formalize this qualitative discussion and describe the SU(N)-generalization of the model, thereby introducing the notations used throughout this paper. In section 3, we proceed directly with the evaluation of the k-string tensions in the SU(N)-version of the model, first in the dilute-plasma approximation, and then with the leading non-diluteness correction taken into account. In section 4, we consider other possible corrections, which stem from the finiteness of the Higgs-boson mass and from the closed electric strings, present in the model. Finally, in section 5, the main results of the paper are summarized, and possible generalizations to the 4-d case are discussed.

2. The model

The euclidean action of the 3-d Georgi-Glashow model reads [15]

$$S = \int d^3x \left[\frac{1}{4g^2} \left(F_{\mu\nu}^a \right)^2 + \frac{1}{2} \left(D_\mu \Phi^a \right)^2 + \frac{\lambda}{4} \left(\left(\Phi^a \right)^2 - \eta^2 \right)^2 \right], \tag{2.1}$$

where the Higgs field Φ^a transforms by the adjoint representation, i.e., $D_\mu \Phi^a \equiv \partial_\mu \Phi^a + \varepsilon^{abc} A_\mu^b \Phi^c$. The weak-coupling regime $g^2 \ll m_W$ parallels then the requirement that η should be large enough to ensure the spontaneous symmetry breaking from SU(2) to U(1). At the perturbative level, the spectrum of the model in the Higgs phase is made of a massless photon, two heavy, charged W-bosons with mass $m_W = g\eta$, and a neutral Higgs field with mass $m_H = \eta\sqrt{2\lambda}$.

What is however more important is the nonperturbative content of the model, represented by the famous 't Hooft-Polyakov monopole [17, 18]. It is a solution to the classical equations of motion, which has the following Higgs- and vector-field parts

- $\Phi^a = \delta^{a3} u(r)$, $u(0) = 0$, $u(r) \stackrel{r\to\infty}{\longrightarrow} \eta - \frac{e^{-m_H r}}{gr}$;
- $A_\mu^{1,2}(\vec{x}) \stackrel{r\to\infty}{\longrightarrow} \mathcal{O}\left(e^{-m_W r}\right)$, $H_\mu \equiv \varepsilon_{\mu\nu\lambda} \partial_\nu A_\lambda^3 = \frac{x_\mu}{r^3} - 4\pi\delta(x_1)\delta(x_2)\theta(x_3)\delta_{\mu 3}$;
- as well as the following action $S_0 = \frac{4\pi\epsilon}{\kappa}$. Here, $\kappa \equiv g^2/m_W$ is the weak-coupling parameter, $\epsilon = \epsilon\left(m_H/m_W\right)$ is a certain monotonic, slowly varying function, $\epsilon \geq 1$, $\epsilon(0) = 1$ (BPS-limit) [19], $\epsilon(\infty) \simeq 1.787$ [20].

The following approximate saddle-point solution (which becomes exact in the BPS-limit) has been found in ref. [15]:

$$S = \mathcal{N} S_0 + \frac{g_m^2}{8\pi} \sum_{\substack{a,b=1 \\ a \neq b}}^{\mathcal{N}} \left(\frac{q_a q_b}{|\vec{z}_a - \vec{z}_b|} - \frac{e^{-m_H |\vec{z}_a - \vec{z}_b|}}{|\vec{z}_a - \vec{z}_b|} \right) + \mathcal{O}\left(g_m^2 m_H e^{-2m_H |\vec{z}_a - \vec{z}_b|} \right) + \mathcal{O}\left(\frac{1}{m_W R} \right),$$

where $m_W^{-1} \ll R \ll |\vec{z}_a - \vec{z}_b|$, $g g_m = 4\pi$, $[g_m] = [\text{mass}]^{-1/2}$. Therefore, while at $m_H \to \infty$, the usual compact-QED action is recovered, in the BPS-limit one has

$$S \simeq \mathcal{N} S_0 + \frac{g_m^2}{8\pi} \sum_{\substack{a,b=1 \\ a \neq b}}^{\mathcal{N}} \frac{q_a q_b - 1}{|\vec{z}_a - \vec{z}_b|},$$

i.e., the interaction of two monopoles doubles for opposite and vanishes for equal charges. Therefore, in this limit, the standard monopole-antimonopole Coulomb plasma recombines itself into two mutually noninteracting subsystems, consisting of monopoles and antimonopoles. The interaction between the objects inside each of these subsystems has a double strength with respect to the interaction in the initial plasma.

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When $m_H < \infty$, the summation over the grand canonical ensemble of monopoles has been performed in ref. [21] and reads

$$\mathcal{Z}_{\text{mon}} = 1 + \sum_{\mathcal{N}=1}^{\infty} \frac{\zeta^{\mathcal{N}}}{\mathcal{N}!} \prod_{a=1}^{\mathcal{N}} \int d^3 z_a \sum_{q_a=\pm 1} \mathrm{e}^{-S} \tag{2.2}$$

$$= \int \mathcal{D}\chi \mathcal{D}\psi \exp\left\{-\int d^3x \left[\frac{1}{2}(\partial_\mu \chi)^2 + \frac{1}{2}(\partial_\mu \psi)^2 + \frac{m_H^2}{2}\psi^2 - 2\zeta \mathrm{e}^{g_m \psi} \cos(g_m \chi)\right]\right\}.$$

Here, χ is the dual-photon field and ψ is the field additional with respect to compact QED, which describes the Higgs boson. Furthermore, the monopole fugacity (i.e., the statistical weight of a single monopole), ζ, has the following form [15]

$$\zeta = \delta \frac{m_W^{7/2}}{g} \mathrm{e}^{-S_0}. \tag{2.3}$$

The function $\delta = \delta\,(m_H/m_W)$ is determined by the loop corrections. It is known [22] that this function grows in the vicinity of the origin (i.e., in the BPS limit). However, the speed of this growth is such that it does not spoil the exponential smallness of ζ in the weak-coupling regime under study.

For the analysis of k-strings in the next sections, we will need the $\mathrm{SU}(N)$-generalization of the partition function (2.2), which reads

$$\mathcal{Z}_{\text{mon}}^N = \int \mathcal{D}\vec{\chi}\mathcal{D}\psi \exp\left[-\int d^3x \left(\frac{1}{2}\left(\partial_\mu \vec{\chi}\right)^2 + \frac{1}{2}\left(\partial_\mu \psi\right)^2 + \frac{m_H^2}{2}\psi^2 - \right.\right.$$

$$\left.\left. - 2\zeta \mathrm{e}^{g_m \psi} \sum_{i=1}^{N(N-1)/2} \cos\left(g_m \vec{q}_i \vec{\chi}\right)\right)\right]. \tag{2.4}$$

It has been taken into account here that, in the $\mathrm{SU}(N)$-case, monopole charges are distributed along the $(N-1)$-dimensional positive root vectors $\vec{q}_i$'s of the group $\mathrm{SU}(N)$ [23], while charges of antimonopoles are represented by roots, that are negative symmetric to those of monopoles. Clearly, the dual-photon field is now described by the $(N-1)$-dimensional vector $\vec{\chi}$. In the next section, we will study k-strings in the compact-QED limit of this theory, while corrections due to the propagation of the Higgs field will be addressed in section 4.

3. k-strings in the weakly coupled $\mathrm{SU}(N)$ 3-d Georgi-Glashow model

In the compact-QED limit, the partition function (2.4) takes the following sine-Gordon–type form:

$$\mathcal{Z}_{\text{mon}}^N = \int \mathcal{D}\vec{\chi} \exp\left[-\int d^3x \left(\frac{1}{2}\left(\partial_\mu \vec{\chi}\right)^2 - 2\zeta \sum_{i=1}^{N(N-1)/2} \cos\left(g_m \vec{q}_i \vec{\chi}\right)\right)\right]. \tag{3.1}$$

Owing to the orthonormality of roots, $\sum_{i=1}^{N(N-1)/2} q_i^\alpha q_i^\beta = \frac{N}{2}\delta^{\alpha\beta}$, $\alpha, \beta = 1, \ldots, N-1$, the following value of the Debye mass of the dual photon stems from eq. (3.1): $m = g_m\sqrt{N\zeta}$.

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The k-string tension is defined by means of the k-th power of the fundamental Wilson loop. The surface-dependent part of the latter is contained in the following expression

$$\left\langle W_k(\mathcal{C})\right\rangle_{\rm mon} = \sum_{a_1,\ldots,a_k=1}^{N} \left\langle \exp\left[-ig\left(\sum_{i=1}^{k}\vec{\mu}_{a_i}\right)\int d^3x \Sigma\left(\vec{x}\right)\vec{\chi}\left(\vec{x}\right)\right]\right\rangle_{\rm mon}, \qquad (3.2)$$

which is a consequence of the formula

$$\mathrm{tr}\,\exp\left(i\vec{\mathcal{O}}\vec{H}\right) = \sum_{a=1}^{N}\exp\left(i\vec{\mathcal{O}}\vec{\mu}_a\right).$$

Here, $\vec{\mathcal{O}}$ is an arbitrary $(N-1)$-component vector, $\vec{H}$ is the vector of diagonal generators and $\vec{\mu}_a$, $a=1,\ldots,N$, are the weight vectors of the fundamental representation of the group SU(N). Furthermore, in eq. (3.2)

$$\Sigma\left(\vec{x}\right) \equiv \int_{\Sigma(\mathcal{C})} d\sigma_\mu\left(\vec{x}(\xi)\right)\partial_\mu^x\delta\left(\vec{x}-\vec{x}(\xi)\right),$$

where $\Sigma(\mathcal{C})$ is an arbitrary surface bounded by the contour $\mathcal{C}$ and parametrized by the vector $\vec{x}(\xi)$, with $\xi = \left(\xi^1,\xi^2\right)$ standing for the 2-d coordinate, $\xi \in [0,1]\times[0,1]$.

The independence of eq. (3.2) of the choice of $\Sigma(\mathcal{C})$ can readily be seen in the same way as for the $(k=1)$-case (see e.g. ref. [24]).[3] The Σ-dependence rather appears in the weak-field, or low-density, approximation, when one may keep only the quadratic term in the expansion of the cosine in eq. (3.1). It has been shown in ref. [24] that the notion "low-density" implies that the typical monopole density is related to the mean one, $\rho_{\rm mean} = \zeta N(N-1)$, by the following sequence of inequalities:

$$\rho_{\rm typical} \ll \zeta\cdot\mathcal{O}(N) \ll \rho_{\rm mean} = \zeta\cdot\mathcal{O}\left(N^2\right).$$

Below in this section, we will discuss in some more details the correspondence between the low-density approximation and the large-N one.

Denoting for brevity $\vec{a} \equiv -g\int d^3x\Sigma\left(\vec{x}\right)\vec{\chi}\left(\vec{x}\right)$, we can rewrite eq. (3.2) as

$$\left\langle W_k(\mathcal{C})\right\rangle_{\rm mon} = \sum_{\substack{a_1,\ldots,a_k=1 \\ \left(\text{with possible repetitions}\right)}}^{N} \left\langle \mathrm{e}^{i\vec{a}\left(\vec{\mu}_{a_1}+\cdots+\vec{\mu}_{a_k}\right)}\right\rangle, \qquad (3.3)$$

where in the low-density approximation the average is defined with respect to the action

$$\int d^3x\left[\frac{1}{2}\left(\partial_\mu\vec{\chi}\right)^2 + \frac{m^2}{2}\vec{\chi}^2\right]. \qquad (3.4)$$

Note that, in this approximation, the string tension for a given k is the same for all surfaces $\Sigma(\mathcal{C})$, which are large enough in the sense $\sqrt{S} \gg m^{-1}$, where S is the area of $\Sigma(\mathcal{C})$ (see the

[3] It is a mere consequence of the quantization condition $gg_m = 4\pi$ and the fact that the product $\vec{\mu}_a\vec{q}_i$ is equal either to $\pm\frac{1}{2}$ or to 0.

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discussion in ref. [24]). In particular, the fundamental string tension reads [24] $\sigma_1 = \frac{N-1}{2N}\bar{\sigma}$, where $\bar{\sigma} \equiv 4\pi^2 \frac{\sqrt{\zeta N}}{g_m}$, and the factor $(N-1)/(2N)$ is the square of a weight vector.

To evaluate eq. (3.3) for $k > 1$, we should calculate the expressions of the form

$$\left(n\vec{\mu}_{a_i} + \sum_{j=1}^{k-n} \vec{\mu}_{a_j}\right)^2 , \tag{3.5}$$

where $(k-n)$ weight vectors $\vec{\mu}_{a_j}$'s are mutually different and also different from the vector $\vec{\mu}_{a_i}$. By virtue of the formula $\vec{\mu}_a\vec{\mu}_b = \frac{1}{2}\left(\delta_{ab} - \frac{1}{N}\right)$, we obtain for eq. (3.5):

$$\frac{N-1}{2N}\left(n^2 + k - n\right) - \frac{1}{2N}\left[2n(k-n) + 2\sum_{l=1}^{k-n-1} l\right] = \frac{k(N-k)}{2N} + \frac{1}{2}\left(n^2 - n\right). \tag{3.6}$$

We should further calculate the number of times a term with a given n appears in the sum (3.5). In what follows, we will consider the case $k < N$, although k may be of the order of N. Then, $C_k^n \equiv \frac{k!}{n!(k-n)!}$ possibilities exist to choose out of k weight vectors n coinciding ones, whose index may acquire any values from 1 to N. The index of any weight vector out of other $(k-n)$ ones may then acquire only $(N-1)$ values, and so on. Finally, the index of the last weight vector may acquire $(N-k+n)$ values. Therefore, the desired number of times, a term with a given n appears in the sum (3.5), reads:

$$\begin{aligned} C_k^n N \cdot (k-n)(N-1) \cdot (k-n-1)(N-2) \cdots 1(N-k+n) &= C_k^n A_N^{k-n+1}(k-n)! \\ &= \frac{k!N!}{n!(n+N-k-1)!}, \end{aligned} \tag{3.7}$$

where $A_N^{k-n+1} \equiv N!/(N-k+n-1)!$. Equations (3.6) and (3.7) together yield for the monopole contribution to the Wilson loop, eq. (3.3):

$$\left\langle W_k(\mathcal{C})\right\rangle_{\text{mon}} = k!N!\mathrm{e}^{-C\bar{\sigma}S} \cdot \sum_{n=1}^{k} \frac{1}{n!(n+N-k-1)!}\mathrm{e}^{-\frac{n^2-n}{2}\bar{\sigma}S}, \tag{3.8}$$

where $C \equiv k(N-k)/(2N)$ is proportional to the Casimir of the rank-k antisymmetric representation of SU(N). We have thus arrived at a Feynman-Kac–type formula, where, in the asymptotic regime of interest, $S \to \infty$, only the first term in the sum is essential. The k-string tension therefore reads $\sigma_k = C\bar{\sigma}$, that yields the Casimir-scaling law (1.1). It is interesting to note that the Casimir of the original unbroken SU(N) group is recovered. This is a consequence of the Dirac quantization condition [23], which distributes the quark charges along the weights of the fundamental representation and the monopole ones along the roots. The orthonormality of the roots then yields the action (3.4), which is diagonal in the dual magnetic variables; the sum of the weights squared is responsible for the Casimir factor, since $C = (\vec{\mu}_{a_1} + \cdots + \vec{\mu}_{a_k})^2$, where all k weight vectors are different from each other. Therefore, terms where all k weight vectors are mutually different yield the dominant contribution to the sum (3.3). Their number in the sum is equal to $k!N!/(N-k)!$, that corresponds to the $(n=1)$-term in eq. (3.8).

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Let us further address the leading correction to the obtained Casimir scaling, which originates from the non-diluteness of plasma. Expanding the cosine on the r.h.s. of eq. (3.1) up to the quartic term, we obtain the action[4]

$$\int d^3x \left[\frac{1}{2}\left(\partial_\mu\vec{\chi}\right)^2+\frac{m^2}{2}\left(\vec{\chi}^2-\frac{g_m^2}{12(N+1)}\vec{\chi}^4\right)\right]. \tag{3.9}$$

By virtue of this formula, one can analyse the correspondence between the $1/N$-expansion and corrections to the low-density approximation. The natural choice for defining the behavior of the electric coupling constant in the large-N limit is the QCD-inspired one, $g=\mathcal{O}(N^{-1/2})$. To make some estimates in this case, let us use an obvious argument that a $\vec{\chi}$-field configuration dominating in the partition function is the one, where every term in the action (3.9) is of the order of unity. When applied to the kinetic term, this demand tells us that the characteristic wavelength l of the field $\vec{\chi}$ is related to the amplitude of this field as $l\sim|\vec{\chi}|^{-2}$. Substituting further this estimate into the condition $l^3m^2|\vec{\chi}|^2\sim 1$, we get $|\vec{\chi}|^2\sim m$. The ratio of the quartic and mass terms, being of the order $|\vec{\chi}|^2g_m^2/N$, can then be estimated as $mg_m^2/N=g_m^3\sqrt{\zeta/N}\sim N\sqrt{\zeta}$. With the exponentially high accuracy, this ratio is small, provided

$$N\lesssim\mathcal{O}\left(\mathrm{e}^{S_0/2}\right). \tag{3.10}$$

Therefore, the non-diluteness corrections are suppressed not at large N, but rather at N's bounded from above by a certain exponentially large parameter.

To proceed with the study of the non-diluteness correction, one needs to solve iteratively the saddle-point equation, corresponding to the average (3.2) taken with respect to the approximate action (3.9). Since it has been demonstrated above that the string tension is defined by the averages $\langle \mathrm{e}^{i\vec{a}(\vec{\mu}_{a_1}+\cdots+\vec{\mu}_{a_k})}\rangle$, where all k weight vectors are mutually different, let us restrict ourselves only to such terms in the sum (3.3). Solving then the saddle-point equation with the Ansatz $\vec{\chi}=\vec{\chi}_0+\vec{\chi}_1$, where $|\vec{\chi}_1|\ll|\vec{\chi}_0|$, we obtain for such a term:

$$-\ln\left\langle \mathrm{e}^{i\vec{a}(\vec{\mu}_{a_1}+\cdots+\vec{\mu}_{a_k})}\right\rangle=\frac{g^2}{2}C\int d^3xd^3y\Sigma\left(\vec{x}\right)D_m\left(\vec{x}-\vec{y}\right)\Sigma\left(\vec{y}\right)+\Delta S\,, \tag{3.11}$$

where $D_m\left(\vec{x}\right)=\mathrm{e}^{-m|\vec{x}|}/(4\pi|\vec{x}|)$ is the Yukawa propagator. The first term on the r.h.s. of eq. (3.11) yields the string tension $\sigma_k=C\bar{\sigma}$, while the second term yields the desired correction. This term reads

$$\begin{aligned}\Delta S&=-\frac{2\pi^2}{3}\frac{(gmC)^2}{N+1}\int d^3x\prod_{l=1}^{4}\left[\int d^3x_lD_m\left(\vec{x}-\vec{x}_l\right)\Sigma\left(\vec{x}_l\right)\right]\\&=-\frac{\pi^2}{6}\frac{(gmC)^2}{N+1}\int d\sigma_{\mu\nu}\left(\vec{x}_1\right)d\sigma_{\mu\nu}\left(\vec{x}_2\right)d\sigma_{\lambda\rho}\left(\vec{x}_3\right)d\sigma_{\lambda\rho}\left(\vec{x}_4\right)\partial_\alpha^{x_1}\partial_\alpha^{x_2}\partial_\beta^{x_3}\partial_\beta^{x_4}I\,,\end{aligned} \tag{3.12}$$

[4]One should use the formula

$$\sum_{i=1}^{N(N-1)/2}q_i^\alpha q_i^\beta q_i^\gamma q_i^\delta=\frac{N}{2(N+1)}\left(\delta^{\alpha\beta}\delta^{\gamma\delta}+\delta^{\alpha\gamma}\delta^{\beta\delta}+\delta^{\alpha\delta}\delta^{\beta\gamma}\right),$$

which stems from the orthonormality of roots.

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where $I \equiv \int d^3x \prod_{l=1}^{4} D_m(\vec{x}-\vec{x}_l)$. The action (3.12) can be represented in the form

$$\Delta S = -\frac{(gC)^2}{(N+1)m^5}\int d\sigma_{\mu\nu}(\vec{x}_1)d\sigma_{\mu\nu}(\vec{x}_2)D(\vec{x}_1-\vec{x}_2)\times$$
$$\times\int d\sigma_{\lambda\rho}(\vec{x}_3)d\sigma_{\lambda\rho}(\vec{x}_4)D(\vec{x}_3-\vec{x}_4)\times G(\vec{x}_1-\vec{x}_3)\,. \tag{3.13}$$

Here, D and G are some positive functions, which depend on $m|\vec{x}_i-\vec{x}_j|$ and vanish exponentially at the distances $\gtrsim m^{-1}$. They can be represented as $D(\vec{x}) = m^4\mathcal{D}(m|\vec{x}|)$, $G(\vec{x}) = m^4\mathcal{G}(m|\vec{x}|)$, where the functions $\mathcal{D}$ and $\mathcal{G}$ are dimensionless.[5]

The derivative expansion yields as leading terms the Nambu-Goto actions:

$$\int d\sigma_{\mu\nu}(\vec{x}_1)d\sigma_{\mu\nu}(\vec{x}_2)D(\vec{x}_1-\vec{x}_2) = \sigma_D\int d^2\xi\sqrt{\mathsf{g}(\vec{x}_1)} + \mathcal{O}\left(\frac{\sigma_D}{m^2}\right). \tag{3.14}$$

Here, $\sigma_D = 2m^2\int d^2z\mathcal{D}(|z|)$ (with z being dimensionless) and $\mathsf{g}(\vec{x}) = \det\|\mathsf{g}^{ab}(\vec{x})\|$ is the determinant of the induced-metric tensor $\mathsf{g}_{ab}(\vec{x}) = (\partial_a\vec{x})(\partial_b\vec{x})$, where $\partial_a \equiv \partial/\partial\xi^a$. Next, the infinitesimal world-sheet element $d\sigma_{\mu\nu}(\vec{x})$ can be represented as $d\sigma_{\mu\nu}(\vec{x}) = \sqrt{\mathsf{g}(\vec{x})}t_{\mu\nu}(\vec{x})d^2\xi$, where $t_{\mu\nu}(\vec{x}) = \varepsilon^{ab}(\partial_a\vec{x})(\partial_b\vec{x})/\sqrt{\mathsf{g}(\vec{x})}$ is the extrinsic-curvature tensor of Σ. We may further take into account that we are interested in the *leading* term of the derivative expansion of the action ΔS, which corresponds to the so short distance $|\vec{x}_1-\vec{x}_3|$, that $t_{\mu\nu}(\vec{x}_1)\,t_{\mu\nu}(\vec{x}_3) \simeq 2$. (Higher terms of the derivative expansion contain derivatives of $t_{\mu\nu}$ and do not contribute to the string tension.) This yields for the integral in eq. (3.13):

$$\frac{\sigma_D^2}{2}\int d\sigma_{\mu\nu}(\vec{x}_1)\,d\sigma_{\mu\nu}(\vec{x}_3)\,G(\vec{x}_1-\vec{x}_3) = \frac{\sigma_D^2}{2}\left[\sigma_G\int d^2\xi\sqrt{\mathsf{g}} + \mathcal{O}\left(\frac{\sigma_G}{m^2}\right)\right],$$

where $\sigma_G = 2m^2\int d^2z\mathcal{G}(|z|)$. We finally obtain from eq. (3.13):

$$\Delta\sigma_k \simeq \frac{(gC)^2\sigma_D^2\sigma_G}{2(N+1)m^5} = \frac{\alpha}{4}\frac{(gC)^2m}{N+1} = \alpha\frac{\bar{\sigma}C^2}{N+1}\,,$$

where α is some dimensionless positive constant. This yields

$$\sigma_k + \Delta\sigma_k = \bar{\sigma}C\left(1+\frac{\alpha C}{N+1}\right).$$

In the limit $k \sim N \gg 1$ of interest, the obtained correction is $\mathcal{O}(1)$, while for $k = \mathcal{O}(1)$ it is obviously $\mathcal{O}(1/N)$. The latter fact enables one to write down the following final result for the leading correction to eq. (1.1) due to the non-diluteness of monopole plasma:

$$R(k,N) + \Delta R(k,N) \equiv \frac{\sigma_k+\Delta\sigma_k}{\sigma_1+\Delta\sigma_1} = C(k,N)\left[1+\alpha\frac{(k-1)(N-k-1)}{2N(N+1)}\right]. \tag{3.15}$$

This expression is as invariant under the replacement $k \to (N-k)$ as the expression (1.1), which does not account for non-diluteness. The fact that at $k \sim N \gg 1$ the obtained correction to the Casimir-scaling law is $\mathcal{O}(1)$, indicates that non-diluteness effects, once being taken into account, can significantly distort the Casimir-scaling behavior.

[5] Our investigations can readily be translated to the Stochastic Vacuum Model of QCD [14] for the evaluation of a correction to the string tension, produced by the four-point irreducible average of field strengths. In that case, the functions D and G would be proportional to the gluonic condensate.

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4. Corrections due to the Higgs field and closed strings

As we have seen in section 2, when one deviates from the compact-QED limit, i.e., makes the Higgs field not infinitely heavy, it starts propagating and opens up an interaction in the monopole plasma via the scalar (rather than vector) component of the monopole solution. The respective partition function is given by eq. (2.4). Averaging in that equation over the Higgs field by means of the cumulant expansion one gets in the second order of this expansion [25]:

$$\mathcal{Z}_{\rm mon}^N \simeq \int \mathcal{D}\vec{\chi} \exp\left[-\int d^3x\left(\frac{1}{2}(\partial_\mu\vec{\chi})^2 - 2\xi\sum_{i=1}^{N(N-1)/2}\cos(g_m\vec{q}_i\vec{\chi})\right) + \right. \tag{4.1}$$
$$\left. + 2\xi^2\int d^3x d^3y \sum_{i,j=1}^{N(N-1)/2}\cos\left(g_m\vec{q}_i\vec{\chi}(\vec{x})\right)\mathcal{K}(\vec{x}-\vec{y})\cos\left(g_m\vec{q}_j\vec{\chi}(\vec{y})\right)\right].$$

In this equation, $\xi \equiv \zeta\exp[\frac{g_m^2}{2}D_{m_H}\left(m_W^{-1}\right)]$ is the modified fugacity (which can be shown to remain exponentially small as long as the cumulant expansion is convergent) and $\mathcal{K}(\mathbf{x}) \equiv \mathrm{e}^{g_m^2 D_{m_H}(\mathbf{x})} - 1$. The Debye mass of the dual photon, stemming from eq. (4.1), reads

$$m = g_m\sqrt{N\xi}\left[1 + \xi I\frac{N(N-1)}{2}\right],$$

where $I \equiv \int d^3x\mathcal{K}(\mathbf{x})$. At $m_H \sim m_W$, the following value of I has been obtained [25]:

$$I \simeq \frac{4\pi}{m_H m_W^2}\exp\left(\frac{4\pi}{\kappa}\mathrm{e}^{-m_H/m_W}\right). \tag{4.2}$$

The parameter of the cumulant expansion is $\mathcal{O}(\xi I N^2)$. By virtue of eqs. (2.3) and (4.2), one can readily see that the condition for this parameter to be (exponentially) small reads $N^2 < \exp[\frac{4\pi}{\kappa}(\epsilon - \frac{1}{\mathrm{e}})]$. Approximating ϵ by its value at infinity, we find

$$N < \mathrm{e}^{8.9/\kappa}. \tag{4.3}$$

Therefore, when one takes into account the propagation of the heavy Higgs boson, the necessary condition for the convergence of the cumulant expansion is that the number of colors may grow not arbitrarily fast, but should rather be bounded from above by some parameter, which is nevertheless exponentially large. A similar analysis can be performed in the BPS limit, $m_H \ll g^2$. There, one readily finds $I \simeq (g_m/m_H)^2$, and

$$\xi I N^2 \propto N^2\exp\left[-\frac{4\pi}{\kappa}\left(\epsilon - \frac{1}{2}\right)\right].$$

Approximating ϵ by its value at the origin, we see that the upper bound for N in this limit is smaller than in the vicinity of the compact-QED limit and reads

$$N < \mathrm{e}^{\pi/\kappa}. \tag{4.4}$$

Note that both constraints (4.3) and (4.4) are more severe, in the respective limits, than the constraint (3.10). Indeed, at $m_H \sim m_W$, eq. (3.10) reads [with the same accuracy as eq. (4.3)] $N \lesssim \mathcal{O}\left(\mathrm{e}^{11.2/\kappa}\right)$, and in the BPS limit it obviously takes the form $N \lesssim \mathcal{O}\left(\mathrm{e}^{2\pi/\kappa}\right)$.

Upon the expansion of cosines in eq. (4.1), one can see that the action (3.9) becomes replaced by

$$\int d^3x \left[\frac{1}{2}\left(\partial_\mu \vec{\chi}\right)^2 + \frac{m^2}{2}\left(\vec{\chi}^2 - \frac{g_m^2}{12(N+1)}\left(1 + 3\xi I N(N+1)\right)\vec{\chi}^4\right)\right].$$

Following further the same steps which led from eq. (3.9) to eq. (3.15), we arrive at the following correction to the latter:

$$R(k,N) + \Delta R(k,N) = C(k,N)\left\{1 + \alpha\frac{(k-1)(N-k-1)}{2}\left[\frac{1}{N(N+1)} + 3\xi I\right]\right\}. \quad (4.5)$$

The obtained correction is therefore exponentially small as long as the parameter of the cumulant expansion is.

Another correction to eq. (1.1) is produced by closed k-strings. Such strings are always present in the sector of a theory, where open k-strings are present (see e.g. ref. [26]). Owing to the above-established fact that the open-string tension is saturated by such configurations where all k weight vectors are mutually different, we may restrict ourselves to consideration of closed strings of the same kind only. Then, according to eq. (3.2), the statistical weight of the interaction of the dual photon with such a closed string reads

$$\exp\left[ig\left(\vec{\mu}_{a_1} + \cdots + \vec{\mu}_{a_k}\right)\int d^3x \Sigma_\mu \partial_\mu \vec{\chi}\right], \quad (4.6)$$

where $\Sigma_\mu(\vec{x},\Sigma) \equiv \int_\Sigma d\sigma_\mu\left(\vec{x}(\xi)\right)\delta\left(\vec{x} - \vec{x}(\xi)\right)$ is the vorticity tensor current defined at the closed-string world-sheet Σ. To model the grand canonical ensemble of closed strings, one can proceed along the lines of ref. [27]. Namely, one can use the fact that these strings are short-living (virtual) objects, whose typical sizes are much smaller than the typical distances between them; therefore, similarly to monopoles in the 3-d Georgi-Glashow model, closed strings form a dilute plasma. The vorticity tensor current of the $\mathcal{N}$-string configuration, i.e. the $\mathcal{N}$-string density, ($\mathcal{N} \geq 1$) reads[6] $\Sigma_\mu^{\mathcal{N}} = \sum_{i=1}^{\mathcal{N}} n_i \Sigma_\mu\left(\vec{x},\Sigma_i\right)$, where n_i's are winding numbers. Similarly to monopoles, only strings with minimal winding numbers, $n_i = \pm 1$, survive, whereas those with $|n_i| > 1$ dissociate into them. The summation over the grand canonical ensemble of closed strings replaces then eq. (4.6) by

$$\sum_{\mathcal{N}=0}^{\infty}\frac{\bar{\zeta}^{\mathcal{N}}}{\mathcal{N}!}\prod_{i=0}^{\mathcal{N}}\int d^3y_i \sum_{n_i=\pm 1}\left\langle \exp\left[ig\left(\vec{\mu}_{a_1} + \cdots + \vec{\mu}_{a_k}\right)\int d^3x \Sigma_\mu^{\mathcal{N}}\partial_\mu \vec{\chi}\right]\right\rangle_{\vec{z}_i(\xi)}. \quad (4.7)$$

Here, $\bar{\zeta}$ is the fugacity of a single closed string, which has the dimensionality $[\mathrm{mass}]^3$, and is an exponentially small quantity in the sense that the mean distance between neighbors in the string plasma, $\mathcal{O}\left(\bar{\zeta}^{-1/3}\right)$, is exponentially large with respect to the characteristic string size. Also, in eq. (4.7), the world-sheet coordinate of the i-th strings has been

[6] We also set by definition $\Sigma_\mu^{\mathcal{N}=0} = 0$.

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split as $\vec{x}_i(\xi) = \vec{y}_i + \vec{z}_i(\xi)$, where the vector $\vec{y}_i = \int d^2\xi \vec{x}_i(\xi)$ describes the position of the string, whereas the vector $\vec{z}_i(\xi)$ describes its shape. Furthermore, the translation- and $O(3)$ invariant measure of integration over string shapes has been denoted as $\langle \ldots \rangle_{\vec{z}_i(\xi)}$. The final expression for the average does not depend on a particular form of this measure. (The only thing which matters is the normalization $\langle 1 \rangle_{\vec{z}_i(\xi)} = 1$.) The result of the average rather does depend on the choice of the UV cutoff, which in the theory under study is, however, unambiguous: $\Lambda = m_W$.

Equation (4.7) produces then the following factor to the r.h.s. of eq. (3.1):

$$\exp\left\{2\bar{\zeta}\int d^3x \cos\left[\frac{g\left(\vec{\mu}_{a_1} + \cdots + \vec{\mu}_{a_k}\right)|\partial_\mu \vec{\chi}|}{m_W^2}\right]\right\}, \tag{4.8}$$

where $\int d^3x$ is nothing else, but the integration over the string position. Also, in eq. (4.8), the absolute value is taken with respect to the Lorentz indices only.[7] In the case of a dilute string plasma under study, we may approximate the cosine in eq. (4.8) by the quadratic term only, that yields the following addendum to the action (3.4):

$$\frac{\bar{\zeta}}{m_W^3}\kappa\int d^3x \left[\left(\vec{\mu}_{a_1} + \cdots + \vec{\mu}_{a_k}\right)\partial_\mu \vec{\chi}\right]^2 .$$

To evaluate with respect to this modified action the average (3.3) (approximated again by a single term where all weight vectors are mutually different), we solve the respective saddle-point equation with the natural Ansatz $\vec{\chi} = \left(\vec{\mu}_{a_1} + \cdots + \vec{\mu}_{a_k}\right)\chi$. The saddle-point equation,

$$\left[-(1+\beta)\partial^2 + m^2\right]\chi = -ig\Sigma, \quad \beta \equiv 2\kappa C\frac{\bar{\zeta}}{m_W^3},$$

due to the smallness of the parameter β, has an approximate solution

$$\chi^{\text{s.p.}}(\vec{x}) \simeq -ig(1-\beta)\int d^3y D_{m_*}(\vec{x}-\vec{y})\Sigma(\vec{y}),$$

where $m_* = m\left(1 - \frac{\beta}{2}\right)$ is the shifted Debye mass. (This shift can be interpreted as an "antiscreening" of the dual photon by closed strings, which carry an *electric* flux and therefore diminish the effect of screening by *magnetic* monopoles.) The string tension accordingly reads [cf. the Casimir-scaling expression $\sigma_k = C\bar{\sigma}$] $\sigma_k = C\left(1 - \frac{3\beta}{2}\right)\bar{\sigma}$, so that eq. (1.1) becomes modified as follows:

$$R(k,N) + \Delta R(k,N) = C(k,N)\left[1 - \frac{3\kappa}{2}\frac{\bar{\zeta}}{m_W^3}\frac{(k-1)(N-k-1)}{N}\right]. \tag{4.9}$$

Although the numerator of the obtained correction is the same as that of eq. (3.15), the denominator is obviously different in the factor $(N+1)$. Consequently, although the correction due to closed strings is small in the factor $\mathcal{O}\left(\kappa\bar{\zeta}/m_W^3\right)$, at $k \sim N$ it scales as $\mathcal{O}\left(N^1\right)$, rather than $\mathcal{O}\left(N^0\right)$, as the correction in eq. (3.15) does. Besides the N-dependence, the obtained correction differs from that of eq. (3.15) by its sign. However,

[7] I.e., $|\partial_\mu \vec{\chi}|^\alpha = \sqrt{\sum_{\mu=1}^3 (\partial_\mu \chi^\alpha)(\partial_\mu \chi^\alpha)}$.

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eq. (4.9) never becomes negative. Indeed, at $k \sim N \gtrsim \mathcal{O}\left(\frac{m_W^3}{\kappa\zeta}\right)$, when this could have happened, the parameter β becomes of the order of unity, i.e. the original approximation $\beta \ll 1$, under which eq. (4.9) has been derived, breaks down. Therefore, at such values of k and N, eq. (4.9) becomes merely invalid.

5. Summary and discussion

In the present paper, we have explored k-string tensions in the SU(N) 3-d Georgi-Glashow model. The advantage of this model for such an analysis is that confinement holds in it at weak coupling and is therefore under a full analytic control. We have first addressed the case of a very dilute monopole plasma and found there the Casimir-scaling behavior for the k-string tension. After that, we have proceeded with the leading non-diluteness correction and found that, at $k \ll N$ it behaves as $\mathcal{O}(1/N)$, whereas at $k \sim N$ this correction is of the order of the leading term. The latter fact means that, in the regime $k \sim N \gg 1$, the non-diluteness effects can significantly change the Casimir-scaling behavior. The non-diluteness correction increases the value of the ratio of string tensions.

We have further addressed two other corrections to the Casimir scaling. One of these is produced by the Higgs-mediated interaction of monopoles, while the other is due to closed k-strings. The Higgs-inspired effect appears as an addendum to the non-diluteness correction and has the same sign as that correction (i.e., increases the ratio of string tensions). However, the effect of the Higgs field is small with respect to the non-diluteness correction, as long as the cumulant expansion, one applies for the average over the Higgs field, is convergent. On the opposite, the correction to the Casimir scaling, produced by closed strings, becomes significant at $k \sim N \gg 1$. Contrary to the non-diluteness and Higgs effects, this correction diminishes the value of the Casimir-scaling ratio.

To conclude, notice that the obtained leading Casimir-scaling behavior may, with a certain care, be translated to the SU(N)-analogue of 4-d compact QED. Obviously, this model can only be viewed as a continuum limit of the respective lattice action [15]. Its crucial difference from the above-discussed 3-d counterpart is that, in this model, confinement holds at strong (rather than weak, as in 3-d) coupling, and instead of the dilute plasma of point-like monopoles one has a non-dilute ensemble of proliferating monopole loops (currents). However, as we have argued in section 3, the non-diluteness corrections to the quadratic dual-photon action are small, provided N is bounded from above by some exponentially large parameter and we will exploit this fact below. One may, nevertheless, take non-diluteness into account by noting that it changes the interaction between monopoles from Coulombic (as in 3-d) to a different type, described by some unknown kernel. On general grounds, one should accept that momenta, transferred by the dual photon in the interactions between monopoles, are conserved. Therefore, the above-mentioned kernel should be translation-invariant, i.e. the interaction of two $\mathcal{N}$-monopole currents,[8] $\vec{j}_\mu^{\mathcal{N}}(x) = g_m \sum_{a=1}^{\mathcal{N}} \vec{q}_{i_a} \oint dx_\mu^a(\tau)\delta\left(x - x^a(\tau)\right)$, should have the form $\vec{j}_\mu^{\mathcal{N}}(x)K(x-y)\vec{j}_\mu^{\mathcal{N}}(y)$.

[8]The magnetic coupling constant, g_m, is obviously dimensionless in four dimensions. Note also that we set by definition $\vec{j}_\mu^{\mathcal{N}=0}(x) = 0$.

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Further, to sum over the grand canonical ensemble of monopoles, let us proceed in the same way as we did for closed strings in the preceeding section. Namely, let us split $x_\mu^a(\tau)$ as $x_\mu^a(\tau) = y_\mu^a + z_\mu^a(\tau)$, where $y_\mu^a = \int_0^1 d\tau x_\mu^a(\tau)$ is the position of the monopole trajectory, whereas the vector $z_\mu^a(\tau)$ describes its shape. Let us also introduce the fugacity of a single-monopole loop, ζ, which has the dimensionality $[\text{mass}]^4$, $\zeta \propto \mathrm{e}^{-S_{\text{mon}}}$. Here, the action of a single a-th loop, obeying the estimate $S_{\text{mon}} \propto g_m^2 \int_0^1 d\tau \sqrt{(\dot{z}_\mu^a)^2}$, is assumed to be of the same order of magnitude for all loops. The summation over the grand canonical ensemble then reads:

$$\sum_{\mathcal{N}=0}^{\infty} \frac{\zeta^{\mathcal{N}}}{\mathcal{N}!} \prod_{a=1}^{\mathcal{N}} \int d^4 y_a \sum_{i_a=\pm 1,\ldots,\pm\frac{N(N-1)}{2}} \left\langle \exp\left(i g_m \vec{q}_{i_a} \oint dz_\mu^a \vec{\chi}_\mu(x_a)\right)\right\rangle =$$

$$= \exp\left[2\zeta \sum_{i=1}^{N(N-1)/2} \int d^4 y \left\langle \cos\left(g_m \vec{q}_i \oint dz_\mu \vec{\chi}_\mu(y+z)\right)\right\rangle\right] \tag{5.1}$$

$$= \exp\left\{2\zeta \sum_{i=1}^{N(N-1)/2} \int d^4 y \sum_{k=1}^{\infty} \frac{(-1)^k}{2k!} \left\langle \left[g_m \vec{q}_i \oint dz_\mu \sum_{n=0}^{\infty} \frac{(z_\nu \partial_\nu^y)^n}{n!} \vec{\chi}_\mu(y)\right]^{2k}\right\rangle\right\},$$

where $\vec{\chi}_\mu$ is the dual-photon field, and $\langle\cdots\rangle$ denotes some rotation- and translation-invariant average over shapes of the loops, $\{z_\mu^a\}$. The term in the action with $k = 1$, $n = 0$ generates the Debye mass of the dual photon. Indeed, this term yields

$$2\zeta \cdot \frac{N}{2} \cdot \frac{g_m^2}{2} \left\langle \oint dz_\mu \oint dz'_\nu \right\rangle \int d^4 y \vec{\chi}_\mu(y) \vec{\chi}_\nu(y),$$

and, due to the rotation and translation invariance of the measure $\langle\ldots\rangle$, the average $\left\langle \oint dz_\mu \oint dz'_\nu \right\rangle$ has the form $L\delta_{\mu\nu}$, where L is some parameter of dimensionality $[\text{length}]^2$. The Debye mass then reads $m = g_m\sqrt{2N\zeta L}$. In the leading low-energy approximation, we may disregard terms with $n \geq 1$ in the sum (5.1). As for the terms with $k \geq 2$ (which account for non-diluteness), as it has been discussed above, they are small and can be disregarded at [cf. eq. (3.10)]

$$N \lesssim \mathcal{O}\left(\mathrm{e}^{S_{\text{mon}}/2}\right). \tag{5.2}$$

Therefore, the leading part of the dual-photon action reads

$$\frac{1}{2} \int d^4 x \vec{\chi}_\mu \left[K^{-1}(x) + m^2\right] \vec{\chi}_\mu,$$

where K^{-1} stands for the inverse operator. This action is the 4-d analogue of the action (3.4), while, as it has already been discussed, the rest of the derivation of eq. (3.8) is entirely based on the properties of weights of the fundamental representation.[9] Therefore, at N bounded from above according to the inequality (5.2), one obtains the Casimir scaling also within the leading low-energy approximation to the SU(N)-analogue of *four-dimensional* compact QED.

[9] The only thing which matters is the generation of the string tension $\bar{\sigma}$ itself, which always takes place as long as the dual-photon propagator, $\left[\tilde{K}^{-1}(p) + m^2\right]^{-1}$, has a massive pole. In the general case, i.e. unless K is adjusted in such a way that at small p^2, $\tilde{K}^{-1}(p) = -m^2 + \ldots$, this indeed holds true.

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References

[1] J. Greensite, *The confinement problem in lattice gauge theory*, *Prog. Part. Nucl. Phys.* **51** (2003) 1 [hep-lat/0301023].

[2] A. Di Giacomo, H.G. Dosch, V.I. Shevchenko and Y.A. Simonov, *Field correlators in QCD: theory and applications*, *Phys. Rept.* **372** (2002) 319 [hep-ph/0007223].

[3] L. Del Debbio and D. Diakonov, *Center vortices and k-strings*, *Phys. Lett.* **B 544** (2002) 202 [hep-lat/0205015].

[4] J. Greensite and Š. Olejník, *k-string tensions and center vortices at large-N*, *J. High Energy Phys.* **09** (2002) 039 [hep-lat/0209088].

[5] J. Ambjørn, P. Olesen and C. Peterson, *Stochastic confinement and dimensional reduction, 1. Four-dimensional* SU(2) *lattice gauge theory*, *Nucl. Phys.* **B 240** (1984) 189.

[6] A. Armoni and M. Shifman, *On k-string tensions and domain walls in $N = 1$ gluodynamics*, *Nucl. Phys.* **B 664** (2003) 233 [hep-th/0304127].

[7] A. Armoni and M. Shifman, *Remarks on stable and quasi-stable k-strings at large-N*, *Nucl. Phys.* **B 671** (2003) 67 [hep-th/0307020].

[8] B. Lucini and M. Teper, *The $k = 2$ string tension in four dimensional* SU(N) *gauge theories*, *Phys. Lett.* **B 501** (2001) 128 [hep-lat/0012025].

[9] B. Lucini and M. Teper, SU(N) *gauge theories in four dimensions: exploring the approach to $N = \infty$*, *J. High Energy Phys.* **06** (2001) 050 [hep-lat/0103027].

[10] B. Lucini and M. Teper, *Confining strings in* SU(N) *gauge theories*, *Phys. Rev.* **D 64** (2001) 105019 [hep-lat/0107007].

[11] L. Del Debbio, H. Panagopoulos, P. Rossi and E. Vicari, *k-string tensions in* SU(N) *gauge theories*, *Phys. Rev.* **D 65** (2002) 021501 [hep-th/0106185].

[12] L. Del Debbio, H. Panagopoulos, P. Rossi and E. Vicari, *Spectrum of confining strings in* SU(N) *gauge theories*, *J. High Energy Phys.* **01** (2002) 009 [hep-th/0111090].

[13] L. Del Debbio, H. Panagopoulos and E. Vicari, *Confining strings in representations with common n-ality*, *J. High Energy Phys.* **09** (2003) 034 [hep-lat/0308012].

[14] V.I. Shevchenko and Y.A. Simonov, *Casimir scaling as a test of QCD vacuum*, *Phys. Rev. Lett.* **85** (2000) 1811 [hep-ph/0001299]; *On Casimir scaling in QCD*, hep-ph/0104135; *Casimir scaling and models of confinement in QCD*, *Int. J. Mod. Phys.* **A 18** (2003) 127; A.I. Shoshi, F.D. Steffen, H.G. Dosch and H.J. Pirner, *Confining QCD strings, Casimir scaling and a euclidean approach to high-energy scattering*, *Phys. Rev.* **D 68** (2003) 074004 [hep-ph/0211287].

[15] A.M. Polyakov, *Quark confinement and topology of gauge fields*, *Nucl. Phys.* **B 120** (1977) 429.

[16] A. Kovner, *Magnetic $Z(N)$ symmetry in $2+1$ dimensions*, *Int. J. Mod. Phys.* **A 17** (2002) 2113 [hep-th/0211248]; *Confinement, magnetic Z(N) symmetry and low-energy effective theory of gluodynamics*, hep-ph/0009138;
A. Kovner and B. Rosenstein, *Topological interpretation of electric charge, duality and confinement in (2+1)-dimensions*, *Int. J. Mod. Phys.* **A 7** (1992) 7419.

[17] G. 't Hooft, *Magnetic monopoles in unified gauge theories*, *Nucl. Phys.* **B 79** (1974) 276.

[18] A.M. Polyakov, *Particle spectrum in quantum field theory*, *JETP Lett.* **20** (1974) 194, reprinted in *Magnetic monopoles*, pp. 103-104, A.S. Goldhaber and W.P. Trower eds., and in *Solitons and particles*, pp. 522-523, C. Rebbi and G. Soliani eds.

[19] M.K. Prasad and C.M. Sommerfield, *An exact classical solution for the 't Hooft monopole and the Julia-Zee dyon*, *Phys. Rev. Lett.* **35** (1975) 760;
E.B. Bogomolny, *Stability of classical solutions*, *Sov. J. Nucl. Phys.* **24** (1976) 449, reprinted in *Solitons and particles*, C. Rebbi and G. Soliani eds., pp. 389-394.

[20] T.W. Kirkman and C.K. Zachos, *Asymptotic analysis of the monopole structure*, *Phys. Rev.* **D 24** (1981) 999.

[21] K. Dietz and T. Filk, *Critical Higgs mass for the* $(2+1)$*-dimensional Georgi-Glashow model*, *Nucl. Phys.* **B 164** (1980) 536.

[22] V.G. Kiselev and K.G. Selivanov, *Quantum correction to monopole mass*, *Phys. Lett.* **B 213** (1988) 165.

[23] S.R. Wadia and S.R. Das, *Topology of quantum gauge fields and duality, 1. Yang-Mills Higgs system in* $(2+1)$*-dimensions*, *Phys. Lett.* **B 106** (1981) 386;
N.J. Snyderman, *The physics of dual vortices and static baryons in* $(2+1)$*-dimensions*, *Nucl. Phys.* **B 218** (1983) 381.

[24] D. Antonov, SU(N) *confining strings*, *J. High Energy Phys.* **09** (2003) 005 [hep-th/0304100].

[25] D. Antonov, *Accounting for the finiteness of the Higgs-boson mass in the 3D Georgi-Glashow model*, *Mod. Phys. Lett.* **A 17** (2002) 279 [hep-th/0201013].

[26] I.I. Kogan and A. Kovner, *Monopoles, vortices and strings: confinement and deconfinement in* $2+1$ *dimensions at weak coupling*, hep-th/0205026.

[27] D. Antonov, *Field correlators in abelian-projected theories and stochastic vacuum model*, *J. High Energy Phys.* **07** (2000) 055 [hep-ph/0006156].

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